WAVEFRONT PROPAGATION THEORY FOR A NONLINEAR LAYERED MEDIUM

M. A. Guzev

UDC 534.2

In [1-3], there is a discussion on wavefront propagation (shock wave or discontinuity in the [2] terminology) in a layered medium. The propagation speed and amplitude have been calculated [3] for a linear medium. Here a nonlinear layered medium is consider for them.

The boundary-value treatment for a nonlinear wave equation is considered. Often, it is convenient to reduce the boundary-value treatment to one concerned with the initial conditions, as they can be useful in a numerical treatment and this is necessary in statistical analysis, since a treatment with initial data provides dynamic causality, which is required in order to construct the statistical theory. The invariant immersion method IIM is often used for that purpose.

A general concept in IIM is that the solution is determined from an immersion system, with a heuristic method of obtaining it as follows: we assume that the treatment is characterized by a parameter and allows an exact solution for a certain initial value of it. Then one varies the parameter to transfer from the simple treatment (soluble exactly) to the actual one, which is related to causality in the equations for the invariant immersion with respect to the adjustable parameter: the solution is governed only by the preceding values of the parameter and is independent of the subsequent ones. That approach has been discussed and used for detailed calculations in various aspects of wave proapagation theory [3-5] and scattering theory [6-8]. Here those equations for the nonstationary case of propagation in a nonlinear layered medium are derived. Those equations are then used to calculate the speed and amplitude.

Let a layer in the medium occupy part of the space $(L_0 \leqslant x \leqslant L)$, on which there is incident from the right a plane wave $\varphi = \varphi(x - L + t)$, which interacts with the medium in such a way that the region x > L produces a reflected wave $R(x - L - t)$, while the wave pattern is $u(x, t) = \varphi(x - L + t) + R(x - L - t)$. The field $u = u(x, t)$ in the layer is defined by

$$u_{tt} - u_{xx} = F_{tt}, \quad F = F(x, u) = \varepsilon(x, u)u, \tag{1}$$

in which $\varepsilon(x, u)$ describes the properties of the medium and the field self-action (the subscripts here and subsequently denote the partial derivatives with respect to the corresponding arguments). Equation (1) arises for example from the description of the electric field of an electromagnetic wave incident from vacuum at right angles to the boundary of a nonmagnetic medium having a nonlinearity of any form [9], in which ε characterizes the deviation of the dielectric constant from one. There is only the transmitted wave $T(x - L_0 + t)$ in the region $x < L_0$. Then u(x, t) satisfies

$$u(x, t) = \varphi(x - L + t) + \int_{L_0}^{L} dx_1 \int dt_1 g F(x_1, u(x_1, t_1))_{t_1 t_1}. \tag{2}$$

Here $g = g(x - x_1, t - t_1) = (1/2)\theta(t - t_1 - |x - x_1|)$ is the Green's function for the wave operating in free space, while the incident field is $\varphi = \int dt_1 g(x - L, t - t_1) f(t_1)$, in which $f(t) = 2\varphi_t(t)$ (2) gives immediately the boundary conditions for (1):

$$(\partial_x + \partial_t) u(x, t)|_{x=L} = f(t), \quad (\partial_x - \partial_t) u(x, t)|_{x=L_0} = 0. \tag{3}$$

Vladivostok. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 63-67, July-August, 1991. Original article submitted October 12, 1989; revision submitted April 6, 1990.

The wave pattern $u(x, t) = u(x, t; L)$ is dependent on L (the position of the right-hand boundary of the layer), and it is used as the immerison parameter. The general IIM concept leads us to relate $\partial u/\partial L$ to the field u. The relationship contains $\partial u/\partial t$ and the variational derivative $\delta u/\delta f$. We vary (2) with respect to L and $f(s_1)$ to get

$$u_L(x, t; L) = \varphi_L(x - L + t) + \int dt_1 g(x - L, t - t_1) F(L, u(L, t_1; L))_{t_1 t_1} + \int\limits_{L_0}^{L} dx_1 \int dt_1 g [F_u(x_1, u) u_L]_{t_1 t_1}, \tag{4}$$

$$\frac{\delta u(x, t; L)}{\delta f(s_1)} = g(x - L, t - s_1) + \int\limits_{L_0}^{L} dx_1 \int dt_1 g \left[F_u(x_1, u) \frac{\delta u}{\delta f(s_1)}\right]_{t_1 t_1}$$

$(u = u(x_1, t_1; L))$. These equations show that the relation between $\delta u/\delta L$ and $\delta u/\delta f$ is to the sought in the form

$$u_L(x, t; L) = \psi(x, t; L) + \int dt_1 \frac{\delta u(x, t; L)}{\delta f(t_1)} F(L, v(t_1, L))_{t_1 t_1} \tag{5}$$

with the unknown functions to be determined $\psi = \psi(x, t; L)$ and $v(t, L) = u(L, t; L)$. On substituting (5) into (4), we see that ψ satisfies an equation in variations for (2) and that the parameter $-t$ is the variation one. Then $\psi = -u_t$ and (5) becomes

$$(\partial_L + \partial_t) u(x, t; L) = \int dt_1 \frac{\delta u(x, t; L)}{\delta f(t_1)} F(L, v(t_1, L))_{t_1 t_1}, \tag{6}$$

which may be considered as a differential equation for the wave field $u(x, t; L)$ with the initial conditions $u(x, t; L)|_{L=x} = u(x, t; x)$. We have $\partial_L v(t, L) = (\partial_L + \partial_x) u(x, t; L)|_{x=L}$ for the unkown function v(t, L). The first term on the right is defined by (6) with x = L, while the second is expressed from (3), so we get the closed integrodifferential equation

$$(\partial_L + 2\partial_t) v(t, L) = f(t) + \int dt_1 \frac{\delta v(t, L)}{\delta f(t_1)} F(L, v(t_1, L))_{t_1 t_1} \tag{7}$$

with initial condition $v(t, L)|_{L=L_0} = \varphi(t)$.

System (6) and (7) is the immersion one for this case and is causal with respect to L; $u(x, t; L)$ at point x is determined by those values of L for which $x \le L$. Further analysis of (6) and (7) may be based on a relation between the variational derivative of the wave field and the derivative with respect to time:

$$\frac{\partial u(x, t; L)}{\partial t} + \int dt_1 f(t_1) \frac{\partial}{\partial t_1} \frac{\delta u(x, t; L)}{\delta f(t_1)} = 0, \tag{8}$$

which follows from (5) and (6) together with the second equation in (4).

The solutions u and v to (6) and (7) are certain functionals of f, which we represent as

$$u = \sum_{n=1}^{\infty} \int \ldots \int dt_1 \ldots dt_n u_n(x, t; t_1, \ldots, t_n; L) \prod_{i=1}^{n} f(t_i),$$
$$v = \sum_{n=1}^{\infty} \int \ldots \int dt_1 \ldots dt_n v_n(t; t_1, \ldots, t_n; L) \prod_{i=1}^{n} f(t_i) \tag{9}$$

with coefficients for u_n and v_n symmetrical with respect to t_i. The time arguments t and t_i appear in the combination $t - t_i$, and to show this, we substitute (9) into (8) and equate the coefficients to powers of f term by term to zero, which gives $\partial_t u_n + \sum_{i=1}^{n} \partial_{t_i} u_n = 0$, whose general solution is an arbitrary function of the first integrals $c_i = t - t_i$ $(i = 1, \ldots, n)$ and variables x and L, so $u_n = u_n(x, t - t_1, \ldots, t - t_n; L)$.

To determine the coefficient functions, we substitute (9) into (6) and (7) and equate the coefficients to powers of f, which gives us a linked equation system. In the linear case, $\varepsilon(x, u) = \varepsilon(x)$, and the only coefficient functions different from zero are $u_1 = G(x, t - t_1; L)$ (the Green's function here) and $v_1 = G(L, t - t_1; L) = H(t - t_1, L)$. The immersion equations

$$(\partial_L + \partial_t) G(x, t - t_1; L) = \varepsilon(L) \int dt_2 G(x, t - t_2; L) H(t_2 - t_1, L)_{t_2 t_2},$$
$$(\partial_L + 2\partial_t) H(t - t_1, L) = \delta(t - t_1) + \varepsilon(L) \int dt_2 H(t - t_2, L) H(t_2 - t_1)_{t_2 t_2}$$

coincide with those obtained in [3, 4]. In general, the system defining the coefficient functions is infinite, and it is necessary to know all those functions in order to derive the wave pattern. It has proved possible to solve this problem in calculating the speed and amplitude.

The wavefront concept is related to the general solution u(x, t) to (1) [1, 2]. We consider a certain curve Γ in the (x, t) space-time for each point on which there exist one-sided bounds to u(x, t), but which are not equal, so u(x, t) has a finite step. Then the motion of point x over Γ can be interpreted as the propagation of the discontinuity in u(x, t), the wavefront, over time. For example, in free space (ε = 0), the general solution $u(x, t) = \theta(x + t)$ to (1) corresponds to a wave whose front propagates with unit velocity and amplitude and which attains the point −x at time t, with curve Γ defined by x = −t.

Now let the medium occupy a half-space (this enables us to avoid considering the reflection at the left-hand boundary), while $\varphi(x - L + t) = (1/2)\theta(x - L + t - t_0)$ in (2) describes a wave whose front attains the boundary x = L at t = t_0. The interaction with the medium results in a reflected wave $R(x - L - t)$ and a shock wave in the medium. The amplitude $\hat{v}_0$ of the shock wave at x = L at t = t_0 + 0 can be derived from (7) with $f(t) = \delta(t - t_0)$, while the speed c and amplitude u_a in the layer are given by (6). For that purpose, one needs to distinguish the singular contributions (~δ functions) and the regular contributions (~θ functions) in (6) and (7). In the first case, we get an equation for c and $\hat{v}_0$ and in the second, a differential equation for u_a.

We first examine the structure of $u = u(x, t; L)$, $u_1 = u_1(x, t; L; s_1) = \delta u/\delta f(s_1)$, $v = v(t, L)$, $v_1 = u_1|_{x=L} = v_1(t, L; s_1)$. We vary (6) and (7) with respect to $f(s_1)$ and put $f = \delta(t - t_0)$, to get an equation for u_1 and v_1, with f on the right in (7) becoming $\delta(t - t_1)$. Then it is clear that v and v_1 contain the factors $\theta(t - t_0)$ and $\theta(t - t_1)$ correspondingly. The L causality in the immersion equations means that

$$u = \theta(t - \sigma(x, L) - t_0)\widehat{u}(x, t; L),$$
$$u_1 = \theta(t - \sigma_1(x, L) - s_1)\widehat{u}_1(x, t; L; s_1), \tag{10}$$

in which σ (σ_1) is the time at which point x is reached by the wavefront u (u_1). We substitute (10) into (8) and equate the coefficients to the δ and θ functions term by term to zero to get

$$\sigma = \sigma_1, \ \widehat{u}(x, \sigma + t_0; L) = \widehat{u}_1(x, \sigma_1 + t_0; L; t_0),$$
$$\widehat{u}_t(x, t; L) = -\widehat{u}_{1t_1}(x, t; L; t_1)|_{t_1=t_0}. \tag{11}$$

The speed c and amplitude $\hat{u}_a$ are defined by $c = 1/\sigma_x$, $u_a = \widehat{u}(x, \sigma + t_0; L)$. To derive σ and $\hat{u}$, we substitute (10) into (6) and (7) and separate the contributions from the δ and θ functions. The system for the δ function coefficients is

$$\sigma_L(x, L) = 1 - F(L, \widehat{v}(t_0, L)),$$
$$2\widehat{v}(t_0, L) = 1 + F(L, \widehat{v}(t_0, L))\widehat{v}_1(t_0, L; t_0). \tag{12}$$

Then (11) and the form of F in (1) give an equation for the shock-wave amplitude $\widehat{v}_0 = \widehat{v}_0(L) = \widehat{v}(t_0, L)$ at x = L and t = t_0 + 0:

$$\varepsilon(L, \widehat{v}_0)\widehat{v}_0^2 - 2\widehat{v}_0 + 1 = 0. \tag{13}$$

In general, there are several solutions $\hat{v}_0$, and the necessary branch is defined by the requirement for continuous linkage with the solution to the linear treatment for $\varepsilon = 0$. We substitute $\widehat{v}_0(L)$ into the first equation in (12) and integrate subject to $\sigma(x, x) = 0$, to get $\sigma(x, L)$. It is clear that $\sigma_x(x, L) = -\sigma_L(x, L)$, and then we use (13) to get the speed in the layers:

$$c = -1/\sqrt{1 - \varepsilon(x, \widehat{v}_0(x))}. \tag{14}$$

The remaining equations after separating the contributions in (6) and (7) are:

$$\begin{aligned}(\partial_L + 2\partial_t)\widehat{v}(t, L) &= \widehat{v}_1(t, L; t_1)\,\partial_{t_1}F(L, \widehat{v}(t_1, L))|_{t_1=t} - \\ &- F(L, \widehat{v}(t_0, L))\,\partial_{t_1}\widehat{v}_1(t, L; t_1)|_{t_1=t_0} - \int_{t_0}^{t} dt_1\,\partial_{t_1}\widehat{v}_1\cdot\partial_{t_1}F;\end{aligned} \tag{15}$$

$$\begin{aligned}(\partial_L + \partial_t)\widehat{u}(x, t; L) &= \widehat{u}_1(x, t; L; t_1)\,\partial_{t_1}F(L, \widehat{v}(t_1, L))|_{t_1=t-\sigma} - \\ &- F(L, \widehat{v}(t_0, L))\,\partial_{t_1}\widehat{u}_1(x, t; L; t_1)|_{t_1=t_0} - \int_{t_0}^{t-\sigma} dt_1\partial_{t_1}\widehat{u}_1\cdot\partial_{t_1}F.\end{aligned} \tag{16}$$

The functions $\widehat{u}_1 = \widehat{u}_1(x, t; L; t_1)$, $\widehat{v}_1 = \widehat{v}_1(t, L; t_1)$, $F = F(L, \widehat{v}(t_1, L))$ appear in the integrals, while the equation for the front amplitude is derived from (16) with $t = \sigma + t_0$. Then the integral on the rhs in (16) becomes zero, and the second term is grouped with $(\partial_L + \partial_t)\widehat{u}$ in du_a/dL. The remaining term on the right contains $\widehat{u}_1(x, \sigma + t_0; L; t_0) = \widehat{u}(x, \sigma + t_0; L)$ (see (11)) and the function $\widehat{v}_t(t, L)|_{t=t_0}$, which we derive from (13) and (15): $\widehat{v}_t(t, L)|_{t=t_0} = -[\partial_L\widehat{v}_0(L)]^2/F_L(L, s)\,s|_{s=\widehat{v}_0(L)}$.

Then u_a satisfies the following equation on the basis of the (1) definition for F:

$$\frac{d}{dL}\ln u_a = -\left(\frac{\partial\widehat{v}_0(L)}{\partial L}\right)^2 \left.\frac{\varepsilon(L, s) + \varepsilon_s(L, s)\,s}{\varepsilon_L(L, s)\,s^2}\right|_{s=\widehat{v}_0(L)} \tag{17}$$

with initial condition $u_a|_{L=x} = \widehat{v}_0(x)$. The r.h.s in (17) is a knwon function of L, so $\ln u_a$ is given by simple quadrature.

For the linear medium, $\varepsilon(x, u) = \varepsilon(x)$, and then $c = -1/\sqrt{1 - \varepsilon(x)}$, and (17) integrates:

$$u_a = \frac{\sqrt{|c(x)\,c(L)|}}{1 + |c(L)|} \tag{18}$$

(this result was obtained in [3]). In the simple case of field self-action, $\varepsilon(x, u) = z(x)u^n \times (|z(x)| \ll 1)$. The $\hat{v}_0$ at $x = L$ at $t = t_0 + 0$ is defined by $z(L)\widehat{v}_0^{n+2} = 2\widehat{v}_0 - 1$, and the necessary branch can be constructed by means of elementary perturbation theory $\widehat{v}_0 = \frac{1}{2} + v(z) = \frac{1}{2} + \sum_{i=1}^{\infty} v_i z^i$, $z = z(L)$. Then we write the rhs for (17) as $-(n+1)zz_L[\partial_z \ln(1 + 2v(z))]^2 \equiv -(n+1)zz_L\sum_{i=0}^{\infty}\varphi_i z^i$ and integrate with respect to L to get

$$u_a = \widehat{v}_0(x)\exp[\Phi(z(x)) - \Phi(z(L))],\quad \Phi(z) = (n+1)\sum_{i=0}^{\infty}\frac{z^{i+2}}{i+2}. \tag{19}$$

If $n = 0$ (linear medium), (19) for u_a coincides with (18), so the essentially new point is the consideration of $n \neq 0$. We see from (19) that u_a is locally dependent on $z(x)$, and in particular if $z(x)$ in (19) is a random function, the deviation of the front amplitude from the value at the boundary $x = L$ in the presence of small fluctuations in z is also a small quantity, so there is no build-up of changes in front amplitude associated with passage through a fluctuating meidum.

The author is indebted to V. I. Klyatskin for suggesting the topic and constant interest.

LITERATURE CITED

1. G. B. Whitham Linear and Nonlinear Waves, Wiley, New York (1974).
2. S. Leibovich and A. Sibass, Nonlinear Waves [Russian translation], Mir Moscow (1977).
3. A. G. Bugrov and V. I. Klyatskin, "The immersion method and the solution of inverse wave treatments for layered media," Izv. Vyssh., Uchebyn. Zaved., Radiofiz., 32, No. 3 (1989).
4. V. I. Klyatskin, The Immersion Method in Wave Propagation Theory [in Russian], Nauka, Moscow (1986).
5. G. I. Babkin, V. I. Klyatskin, and L. Ya. Lyubavin, "Boundary-value problems for the wave equation," Akust. Zh. 28, No. 1 (1982).
6. F. Calogero, The Phase-Function Method in Potential-Scattering Theory [Russian translation], Mir, Moscow (1972).
7. V. V. Babikov, The Phase-Function Method in Quantum Mechanics [in Russian], Nauka, Moscow (1988).
8. D. I. Abramov, "Recovering an inteaction potential from scattering data by means of nonlinear equations," Dokl. Akad. Nauk SSSR, 298, No. 3, (1988).
9. N. Blombergen, Nonlinear Optics [Russian translation], Mir, Moscow (1966).